Shoelace Method in Higher Dimensions

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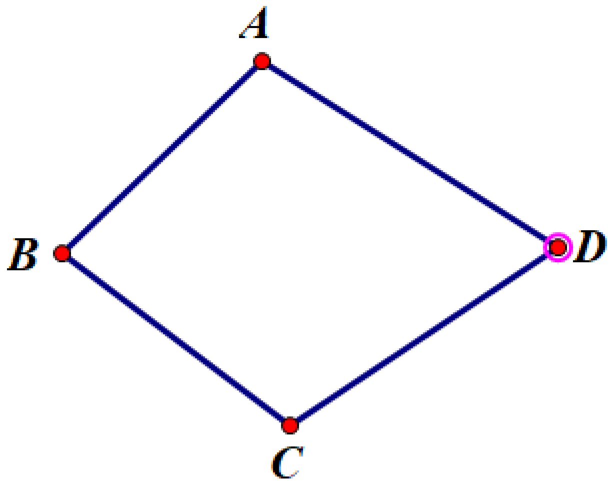
Introduction

Shoelace Method in 2D

$$\frac{1}{2}\begin{vmatrix} x_A & x_B & x_C & x_D & x_A \\ y_A & y_B & y_C & y_D & y_A \end{vmatrix}$$



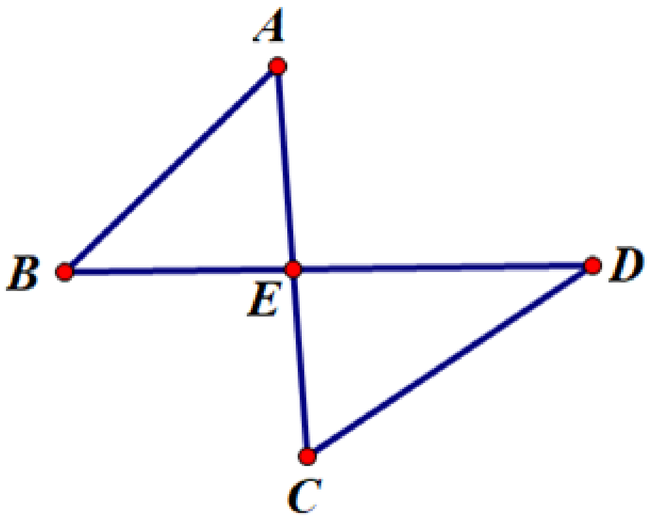
$$= \frac{1}{2}(x_A y_B + \mathrm{x_B} y_C + x_C y_D + x_D y_A - x_B y_A - x_C y_B - x_D y_C - x_A y_D)$$

Shoelace Method in 2D

$$\frac{1}{2}\begin{vmatrix} x_A & x_C & x_D & x_B & x_A \\ y_A & y_C & y_D & y_B & y_A \end{vmatrix}$$

$$= \frac{1}{2}(x_A y_C + x_C y_D + x_D y_B + x_B y_A - x_C y_A - x_D y_C - x_B y_D - x_A y_B)$$

$$= S_{ABE} - S_{\mathrm{CDE}}$$



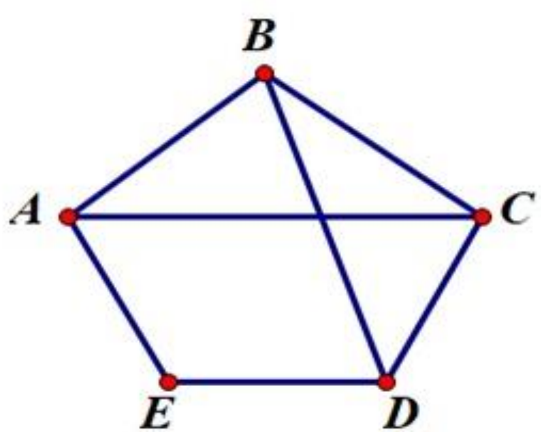
Motivation and Objective

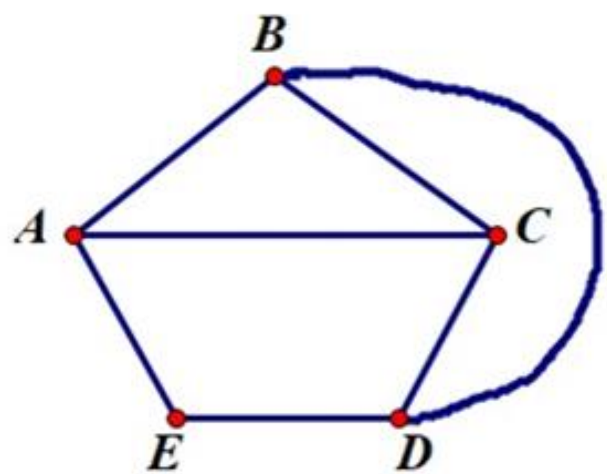
- Motivation
  - Previous Project
  - Interest in exploration
- Objective
  - I aim to establish the relationship between the result of “shoelace method” on 3-dimension figure to its volume.

Graph Theory Preliminaries

Vertices and Edges of Graphs

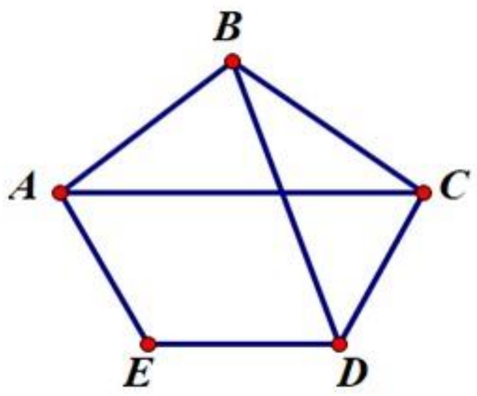
- Vertices (Points): A, B, C, D, E
- Edges (line segments): AB, AC, BD, etc.
- Relative positions of vertices NOT important! Only Adjacency of vertices are important.
- The two graphs below are equivalent (isomorphic)





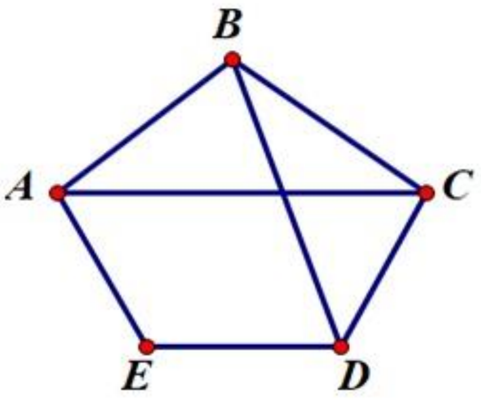
Walks

- Incidence of vertices and edges
- Walk: an **alternating** sequence of **vertices and edges**, consecutive elements of which must be **incident**. It starts and ends with vertices.
- To simplify the representation of walks, we use only the vertices in the sequence.
- Example: ACD, ABCDED, ABDCA etc.



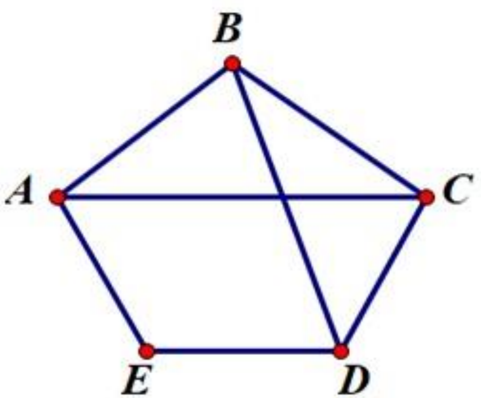
Walks

- Length of a walk is the same as the number of edges in that walk.
- Example: ACD has a length of 2, ABCDED has a length of 5, and ABDCA has a length of 4



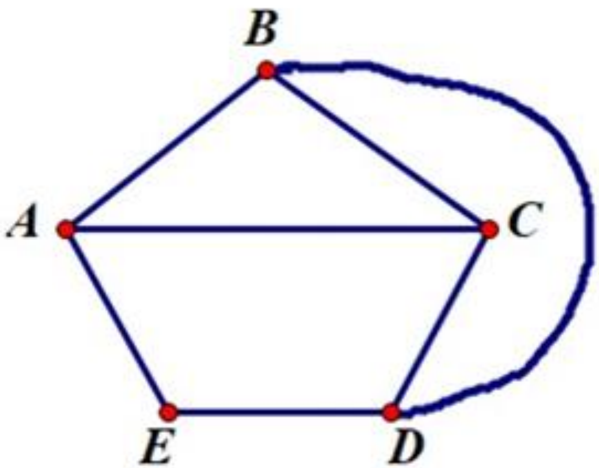
Walks

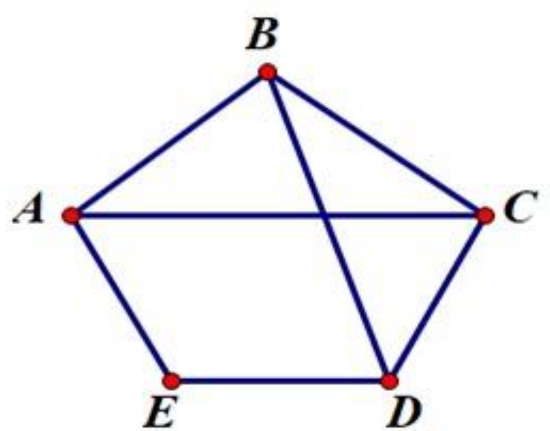
- Closed walks are walks with the starting and ending vertex identical.
- Example: ACD and ABCDED are not closed walks, but ABDCA is a closed walk.



Planar Graphs and Their Faces

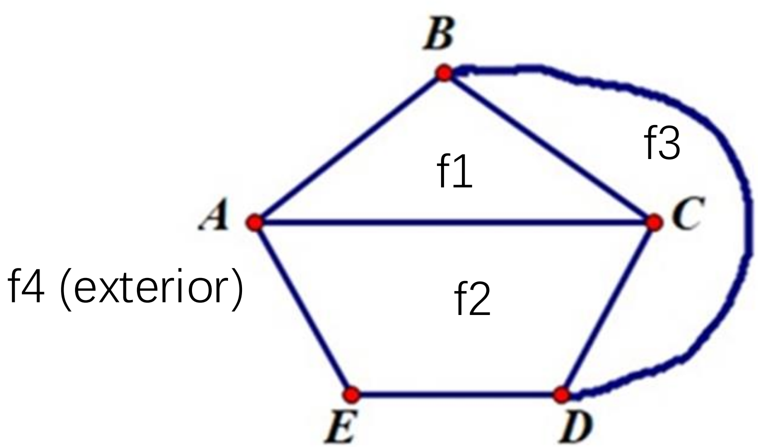
- On a planar embedding, **no two edge cross each other.**
- If there exist planar embeddings that are isomorphic to G, then G is a planar graph.
- All planar graphs mentioned from now on will be referring to their planar embeddings.





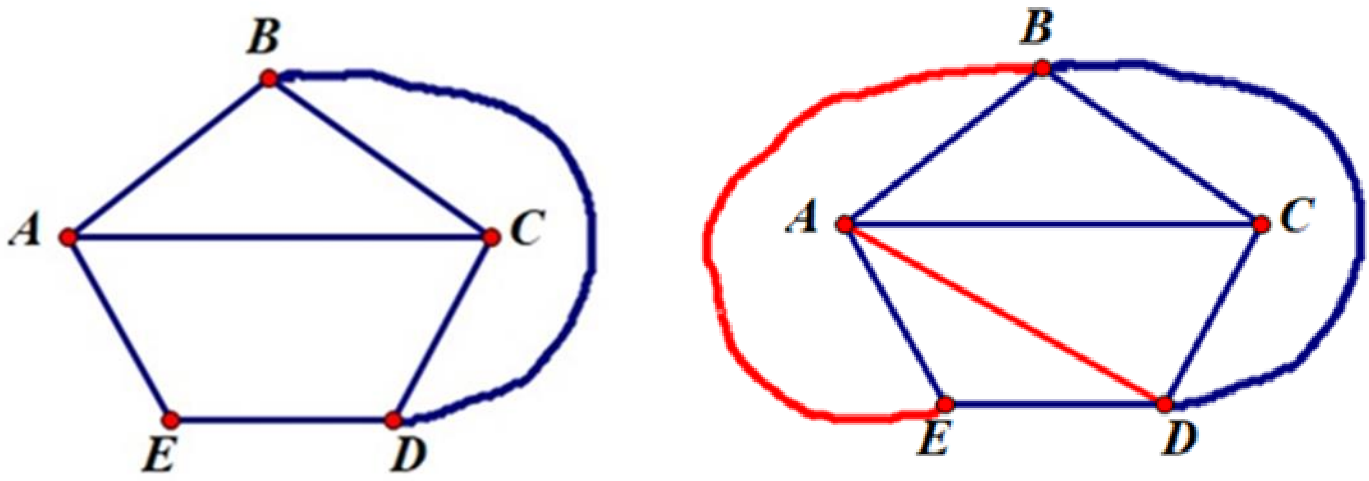
Planar Graphs and Their Faces

- Faces refers to different parts of the plane which is separated by the edges of the plane graph.
- The set of all faces in of a certain planar graph should fill the entire plane.
- Note: parts that can be further broken down are not considered a face. For example, ABCDE is not a face.



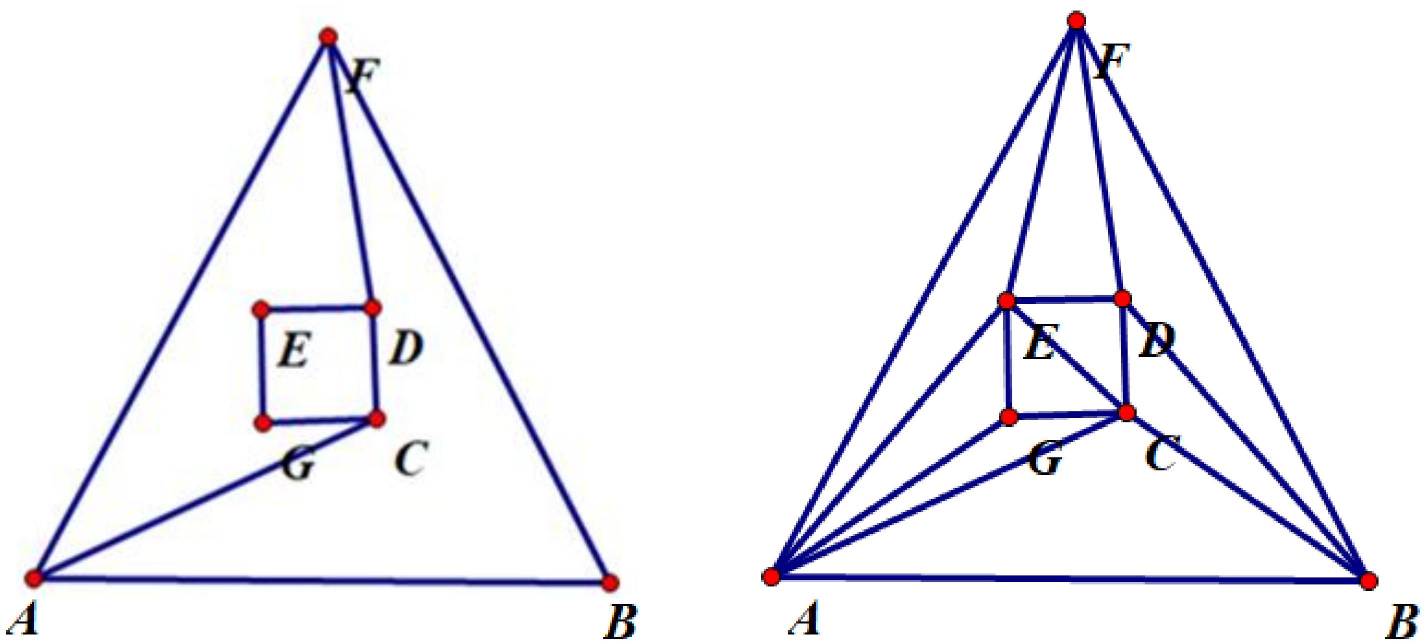
Maximization of Planar Graphs

- Maximizing a planar graph is equivalent to adding edges between previously non-connected vertices, until the point where adding any new edge will make the graph a non-planar graph.



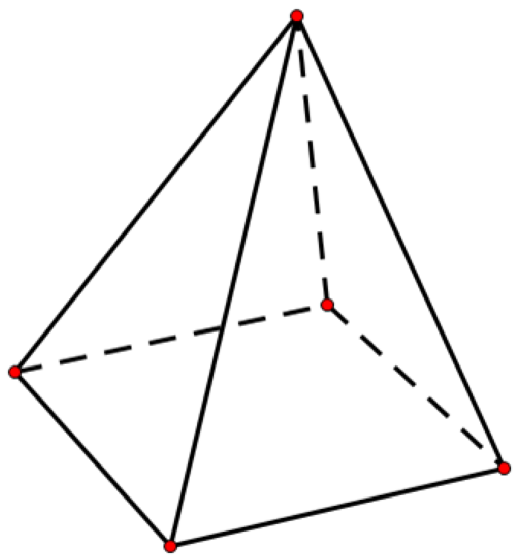
Maximal Planar Graphs(MPGs)

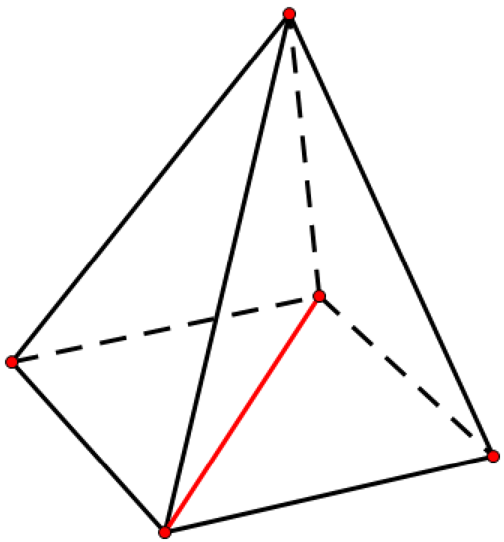
- Maximal planar graphs are planar graphs which has been maximized. Below is a more complicated example.



Triangulation of Polyhedron

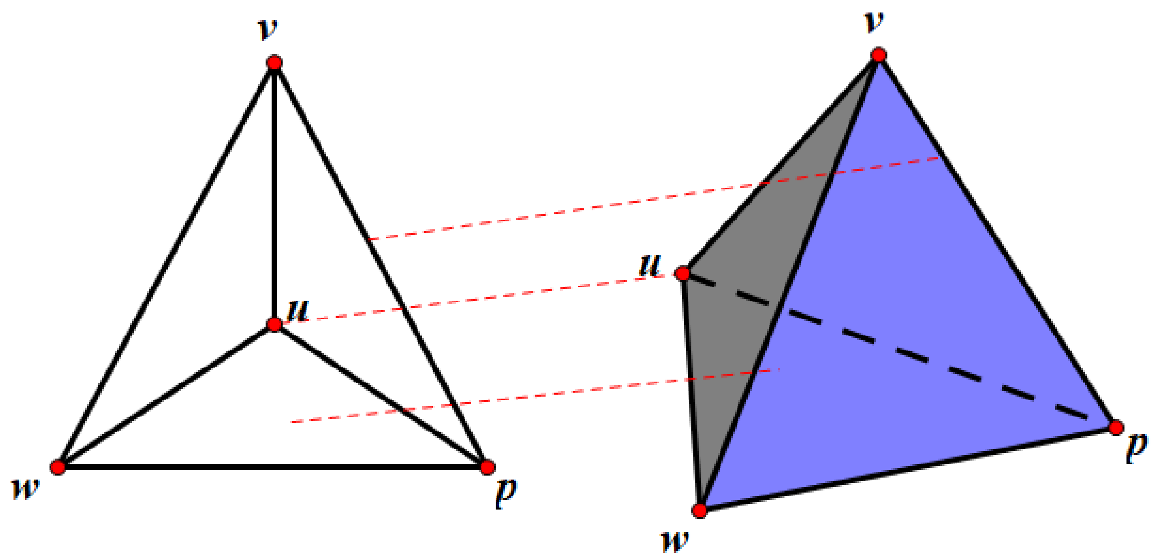
- When triangulating a convex polyhedron, we simply triangulate each face of the polyhedron.





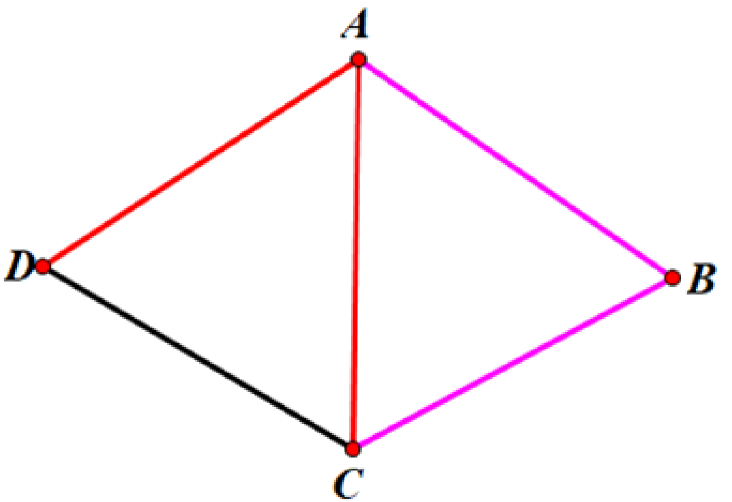
Steinitz's Theorem

- Each simplicial convex polyhedron corresponds to a maximal planar graph and each maximal planar graph can be constructed into a simplicial convex polyhedron.



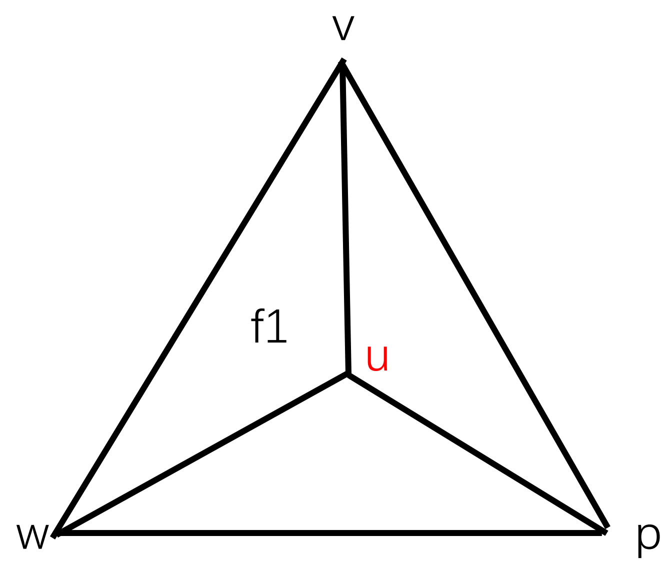
Consecutive Edge Pairs (CEPs)

- A CEP is a pair of adjacent edges on an MPG that corresponds to a face. (None directed)



The Petrie Walk

1. Start with any face $f1$ and any vertex u in f_1.

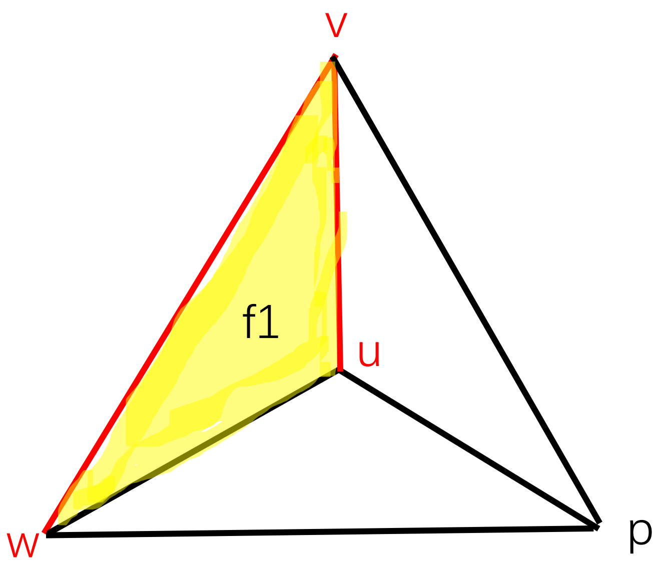


V-Representation:
E-Representation:

The Petrie Walk

1. Start with any face $f1$ and any vertex u in f_1.

2. Starting from u, trace a CEP in $f1$, in **anti-clockwise** manner. This CEP can be represented as **{uv, vw}**



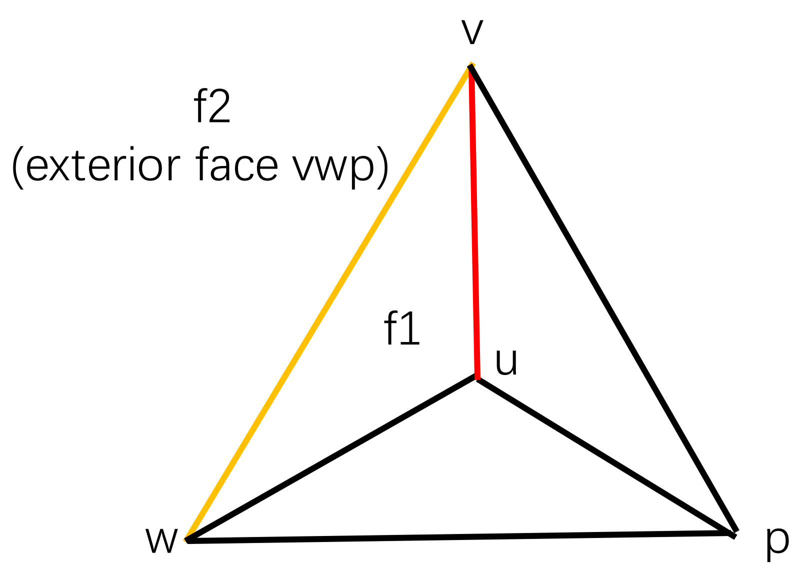
V-Representation: u v w
E-Representation: [uv, vw]
Note: The square bracket includes the two edges in the current CEP

The Petrie Walk

1. Start with any face $f1$ and any vertex u in f_1.

2. Starting from u, trace a CEP in $f1$, in anti-clockwise manner. This CEP can be represented as {uv, vw}

3. Identify the face that share the second edge of the previous CEP, and name it $f2$.



V-Representation: u v w
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Note: The square bracket includes the two edges in the current CEP

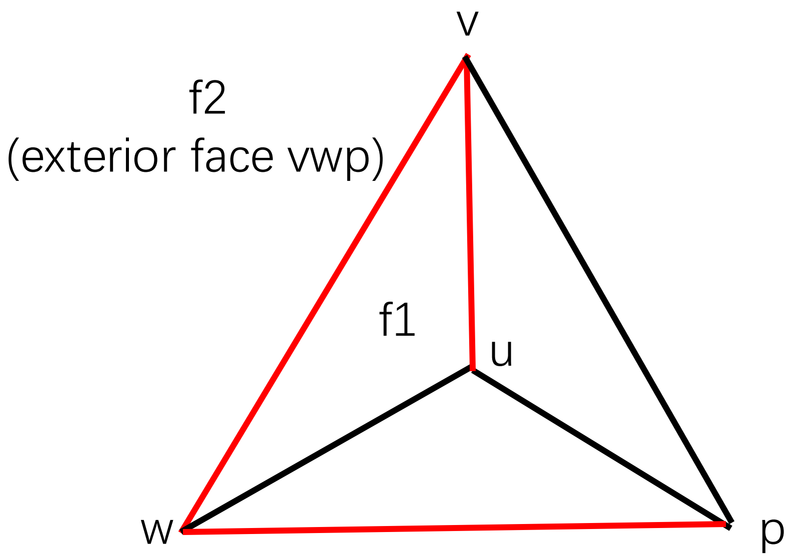
The Petrie Walk

1. Start with any face $f1$ and any vertex u in f_1.

2. Starting from u, trace a CEP in $f1$, in anti-clockwise manner. This CEP can be represented as {uv, vw}

3. Identify the face that share the second edge of the previous CEP, and name it $f2$.

4. Trace a new CEP in f_2, starting with the second edge of the previous CEP.



V-Representation: u v w p
E-Representation: uv, [vw, wp]
Note: The square bracket includes the two edges in the current CEP

The Petrie Walk

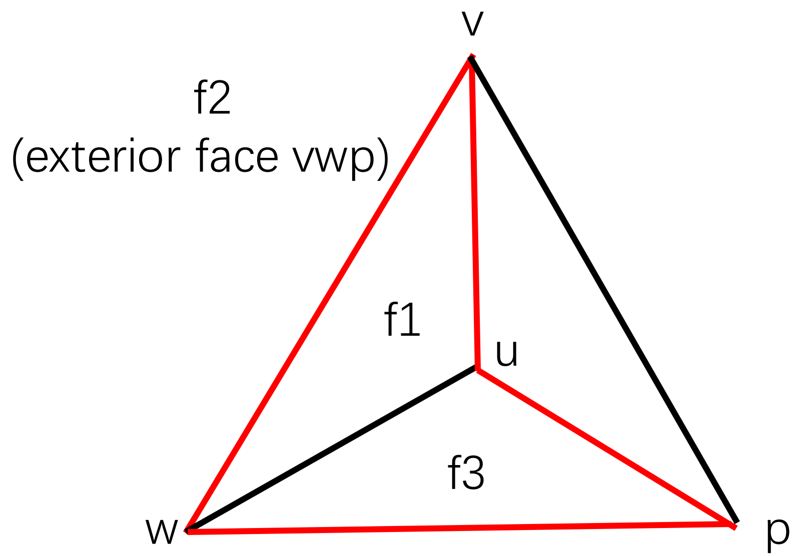
1. Start with any face $f1$ and any vertex u in f_1.

2. Starting from u, trace a CEP in $f1$, in anti-clockwise manner. This CEP can be represented as {uv, vw}

3. Identify the face that share the second edge of the previous CEP, and name it $f2$.

4. Trace a new CEP in f_2, starting with the second edge of the previous CEP.

5. **Repeat step 3,4** in the newly found face f_3



V-Representation: u v w p u

E-Representation: uv, vw, [wp, pu]

Note: The square bracket includes the two edges in the current CEP

The Petrie Walk

1. Start with any face $f1$ and any vertex u in f_1.

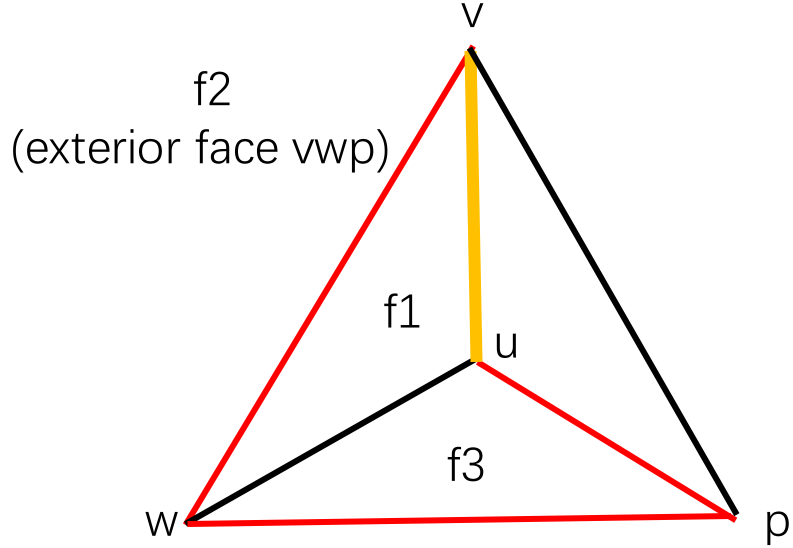
2. Starting from u, trace a CEP in $f1$, in anti-clockwise manner. This CEP can be represented as {uv, vw}

3. Identify the face that share the second edge of the previous CEP, and name it $f2$.

4. Trace a new CEP in f_2, starting with the second edge of the previous CEP.

5a. **Repeat step 3,4** in the newly found face f_3

5b. The process stops **until we trace CEP {uv,vw} again.** However, the entire walk terminates at u.



V-Representation: u v w p u
E-Representation: uv, vw, [wp, pu]
Note: The square bracket includes the two edges in the current CEP

The Petrie Walk

1. Start with any face $f1$ and any vertex u in f_1.

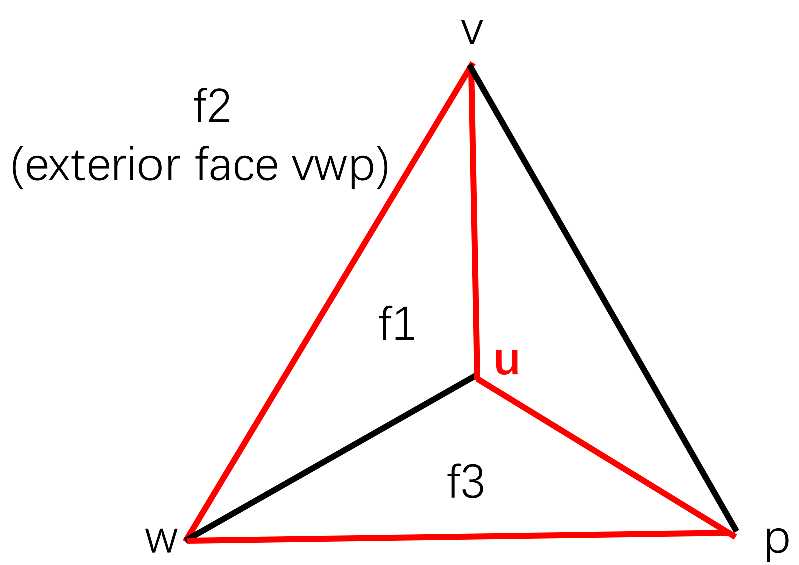
2. Starting from u, trace a CEP in $f1$, in anti-clockwise manner. This CEP can be represented as {uv, vw}

3. Identify the face that share the second edge of the previous CEP, and name it $f2$.

4. Trace a new CEP in f_2, starting with the second edge of the previous CEP.

5a. **Repeat step 3,4** in the newly found face f_3

5b. The process stops **until we trace CEP {uv,vw} again.** However, the entire walk terminates at u. Note that a Petrie Walk always starts and ends at the same vertice.



V-Representation: u v w p u
E-Representation: uv, vw, [wp, pu]
Note: The square bracket includes the two edges in the current CEP

The Petrie Walk

Examples:

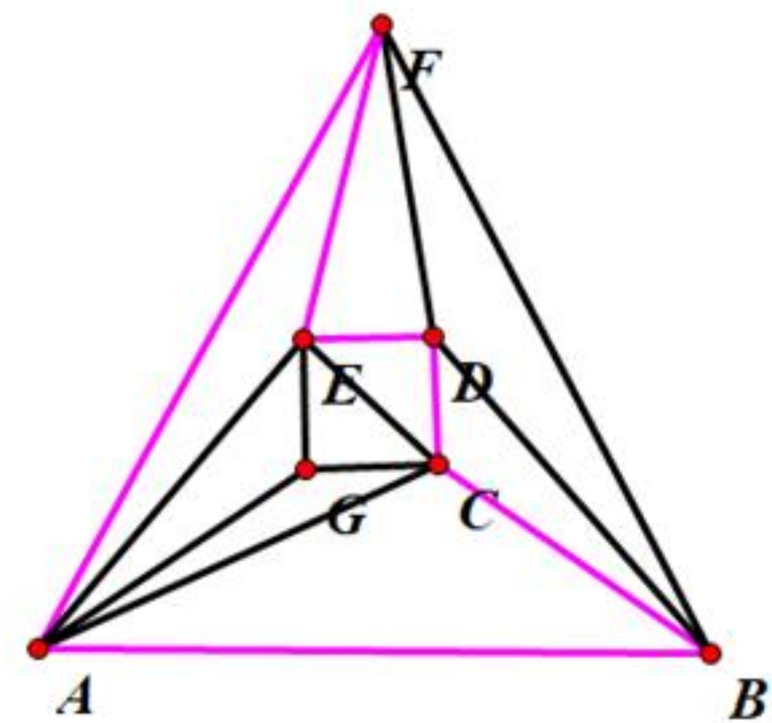


Figure 4.4
w1: ABCDEF(A)

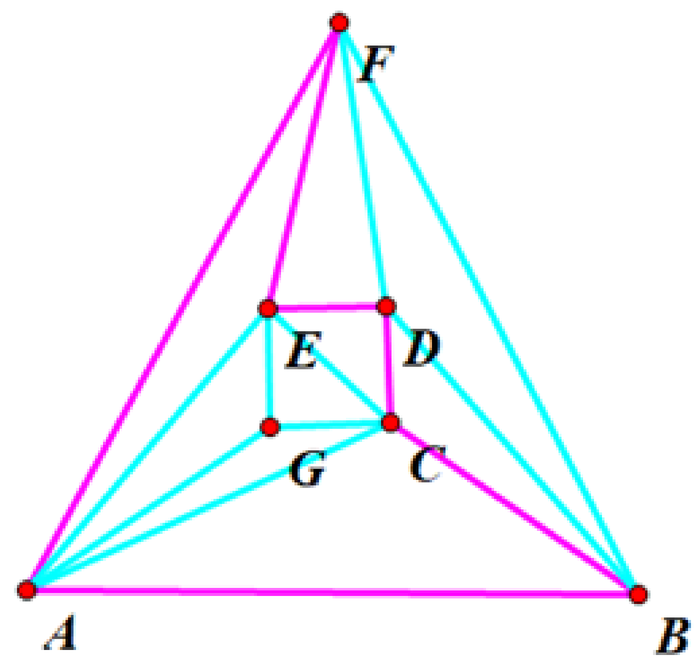


Figure 4.5
w2: CABFDECGAEFDBCAGECDBFAED(C)

Equality of two Petrie Walks

- Suppose w1 and w2 are two Petrie Walk, they are the same if and only if they contain a common Consecutive Edge Pair. Other Wise, they are distinct.

W1: 12341

W2: 23412

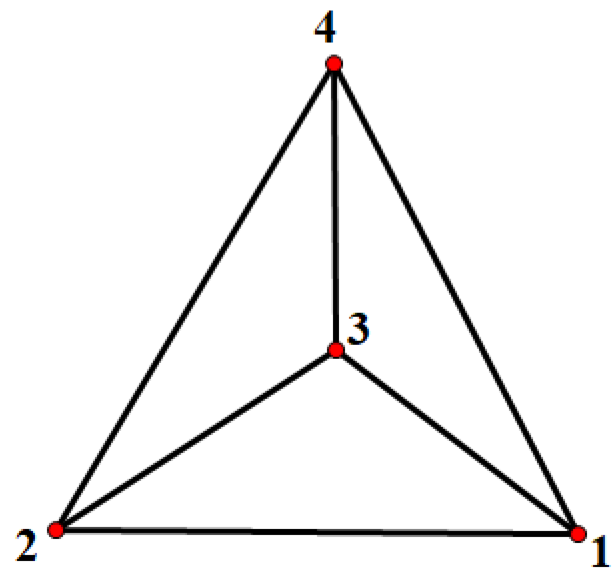
W3: 43214

W4: 24132

W1 and W2: Same

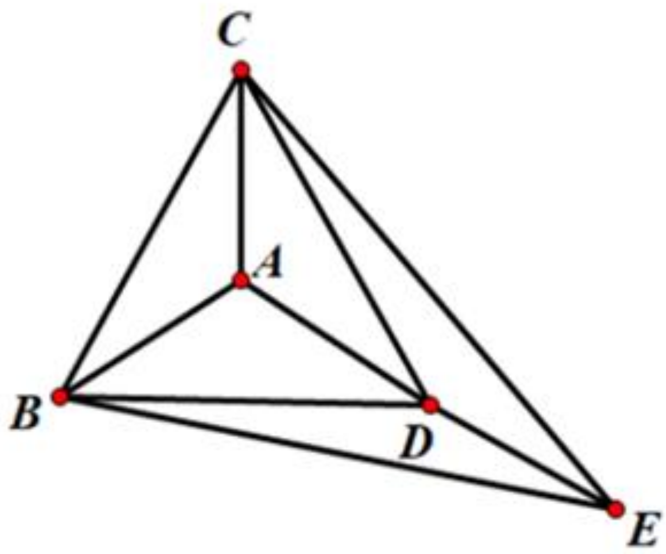
W2 and W3: Same

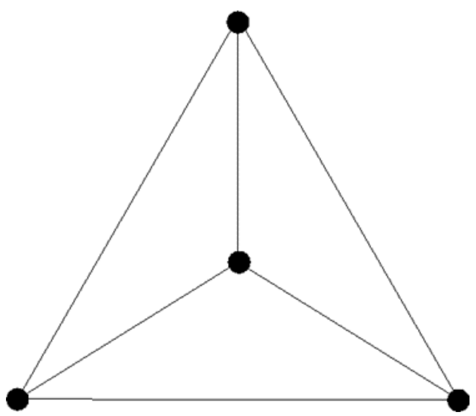
W3 and W4: Distinct



Counting Petrie Walks

- According to the criteria, the MPG of bipyramid on the left has only one Petrie Walk while the MPG of a tetrahedron have three distinct Petrie Walks

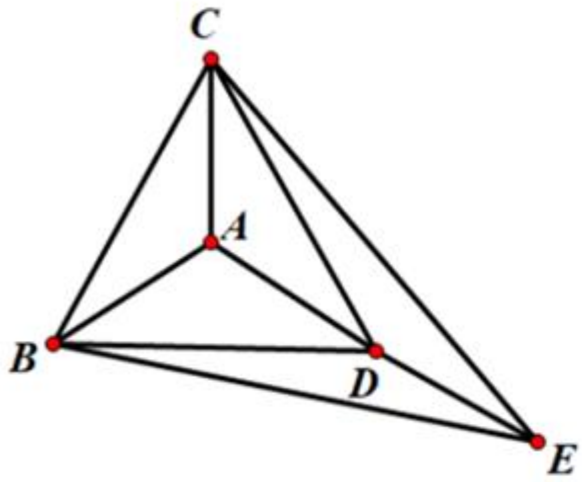


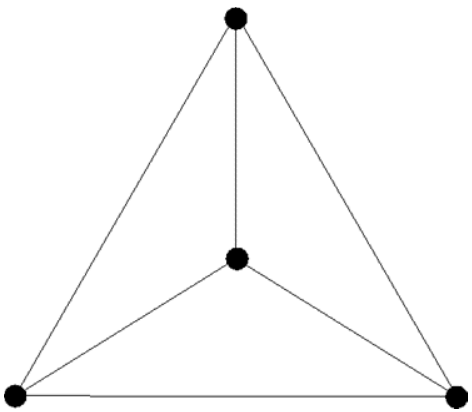


PCMPGs

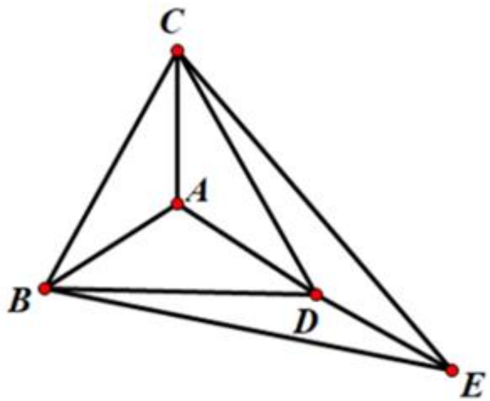
Petrie Complete MPGs (PCMPGs):

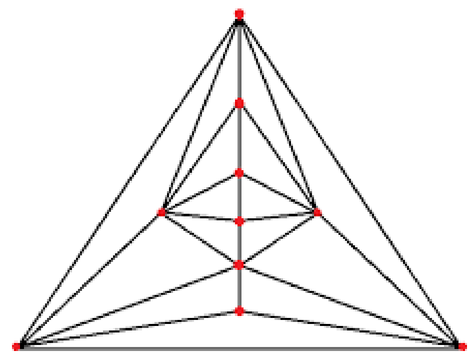
- An MPG is Petrie Complete if and only if G only contains one distinct Petrie walk.
- The graph on the left is a PCMPG but the graph on the right is not.

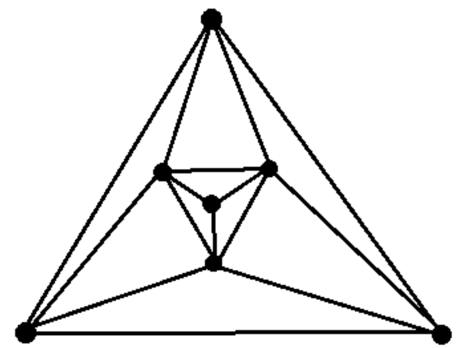




Verification of PCMPG







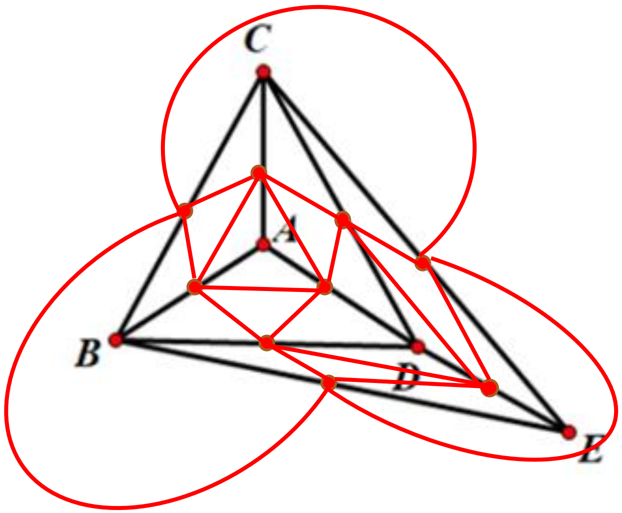
PCMPG?

The length of the Petrie Walk of a PCMPG always equal to 6n-12, where n is the number of vertices of the graph.
Furthermore, we have also proven that even on a non-PC MPG, the sum of length of all distinct Petrie Walks still equal to 6n-12

Medial Graphs

Medial Graphs:

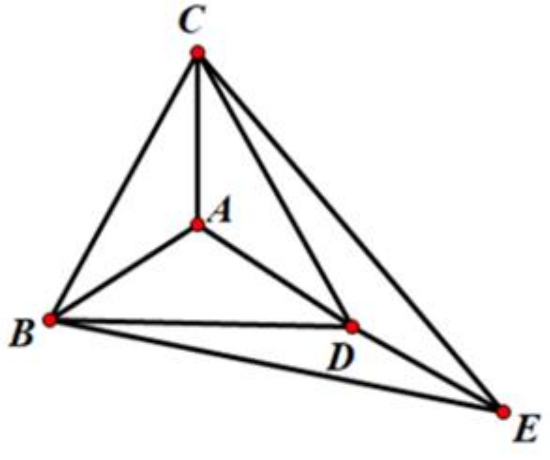
A medial graph of an MPG, G, is denoted as Med(G). Med(G) has a vertex wherever G has an edge, and two vertices in Med(G) are adjacent if and only if the two corresponding edges in G are of a common face. The diagram below shows an MPG (in black) and its medial graph (in red).

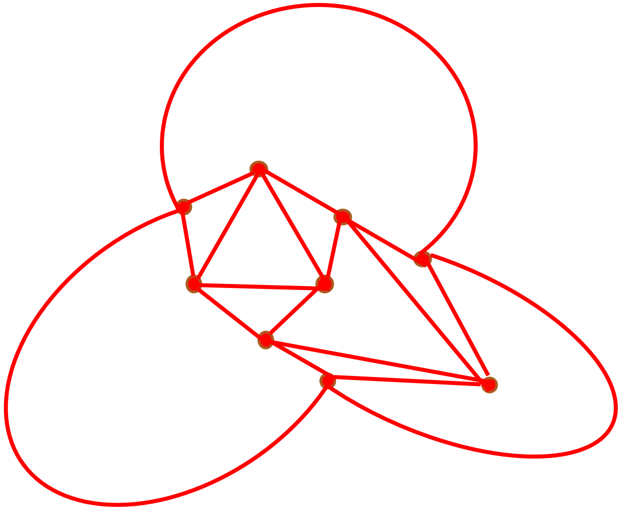


PCMPGs and Medial Graphs

Proposition: Let G be an MPG. A Petrie Walk in G corresponds to an Eulerian circuit in Med(G) if and only if G is a PCMPG.

The example below demonstrates this proposition.





Extended Shoelace Method in 3D

Defining Shoelace Multiplication

$$\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} = x_1 y_2 - x_2 y_1$$

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} = x_1 y_2 z_3 - x_3 y_2 z_1$$

Polarity

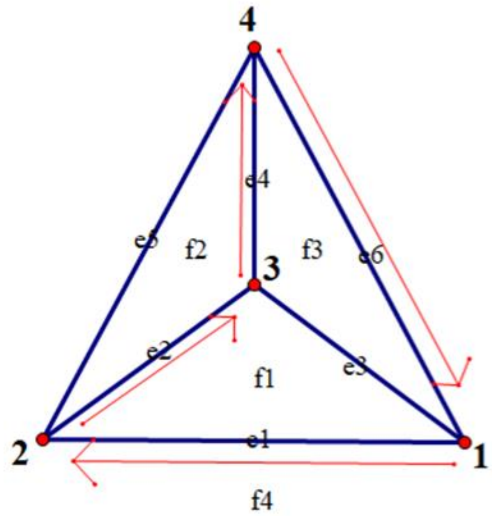
$$\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}^{+} = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} = x_1 y_2 z_3 - x_3 y_2 z_1$$

$$\begin{vmatrix} x_3 & x_2 & x_1 \\ y_3 & y_2 & y_1 \\ z_3 & z_2 & z_1 \end{vmatrix}^{-} = \begin{vmatrix} x_3 & x_2 & x_1 \\ y_3 & y_2 & y_1 \\ z_3 & z_2 & z_1 \end{vmatrix} = x_1 y_2 z_3 - x_3 y_2 z_1$$

Volume Computation

w_1: 12341(2)

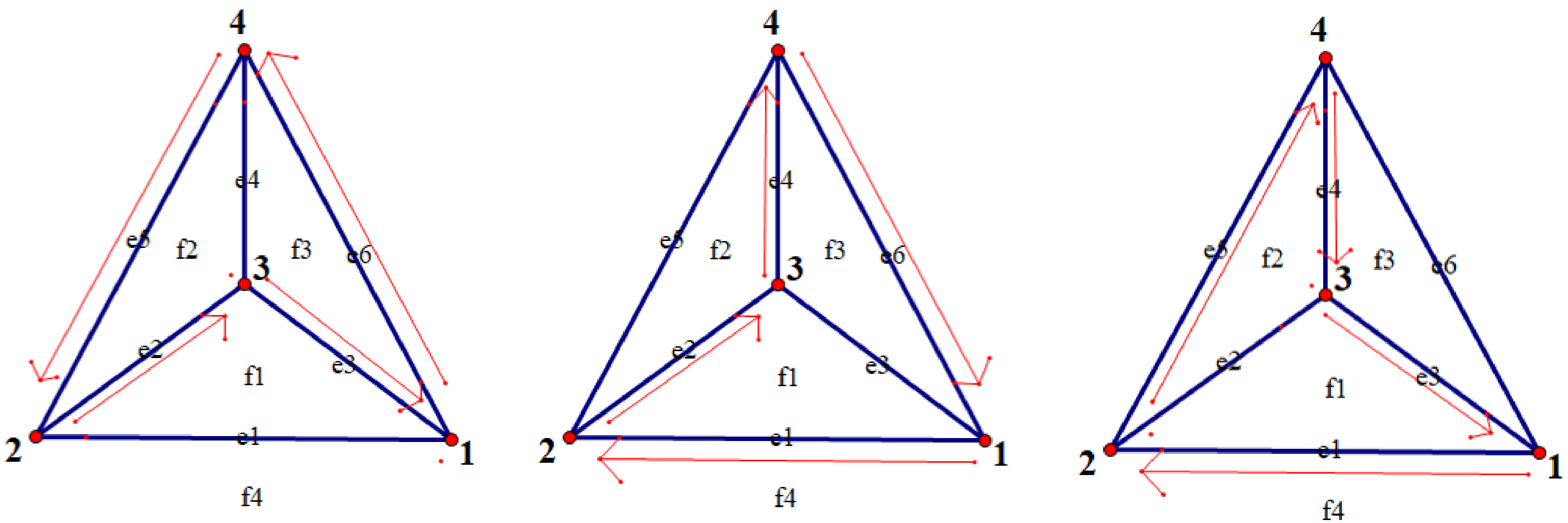
Second edge of the last CEP must be added, corresponding to one vertex added in the Shoelace Formula



$$\begin{vmatrix} x_1 & x_2 & x_3 & x_4 & x_1 & x_2 \\ y_1 & y_2 & y_3 & y_4 & y_1 & y_2 \\ z_1 & z_2 & z_3 & z_4 & z_1 & z_2 \end{vmatrix}$$

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}^{-} + \begin{vmatrix} x_2 & x_3 & x_4 \\ y_2 & y_3 & y_4 \\ z_2 & z_3 & z_4 \end{vmatrix}^{+} + \begin{vmatrix} x_3 & x_4 & x_1 \\ y_3 & y_4 & y_1 \\ z_3 & z_4 & z_1 \end{vmatrix}^{-} + \begin{vmatrix} x_4 & x_1 & x_2 \\ y_4 & y_1 & y_2 \\ z_4 & z_1 & z_2 \end{vmatrix}^{+}$$

Shoelace Multiplication and determinant

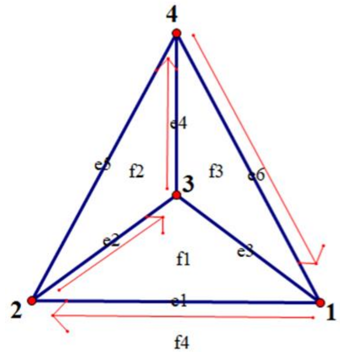


The CEPs corresponding to f_3 are:

$$\begin{vmatrix} x_3 & x_1 & x_4 \\ y_3 & y_1 & y_4 \\ z_3 & z_1 & z_4 \end{vmatrix}^{+} + \begin{vmatrix} x_3 & x_4 & x_1 \\ y_3 & y_4 & y_1 \\ z_3 & z_4 & z_1 \end{vmatrix}^{-} + \begin{vmatrix} x_4 & x_3 & x_1 \\ y_4 & y_3 & y_1 \\ z_4 & z_3 & z_1 \end{vmatrix}^{+} = \det\begin{pmatrix} x_3 & x_1 & x_4 \\ y_3 & y_1 & y_4 \\ z_3 & z_1 & z_4 \end{pmatrix}$$

$$= x_3 y_1 z_4 - x_4 y_1 z_3 + x_1 y_4 z_3 - x_3 y_4 z_1 + x_4 y_3 z_1 - x_1 y_3 z_4$$

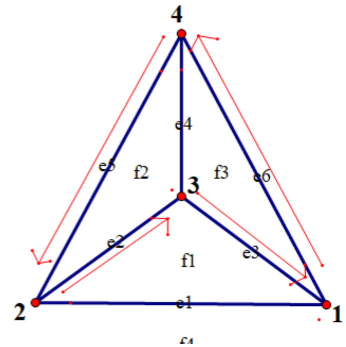
Volume Computation



$$\begin{vmatrix} x_1 & x_2 & x_3 & x_4 & x_1 & x_2 \\ y_1 & y_2 & y_3 & y_4 & y_1 & y_2 \\ z_1 & z_2 & z_3 & z_4 & z_1 & z_2 \end{vmatrix}$$

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}^- \quad \begin{vmatrix} x_2 & x_3 & x_4 \\ y_2 & y_3 & y_4 \\ z_2 & z_3 & z_4 \end{vmatrix}^+$$

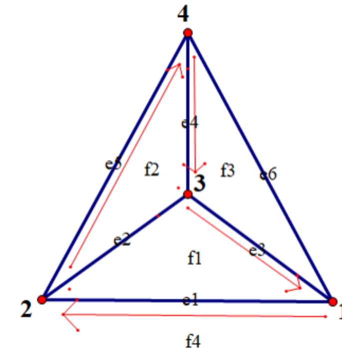
$$\begin{vmatrix} x_3 & x_4 & x_1 \\ y_3 & y_4 & y_1 \\ z_3 & z_4 & z_1 \end{vmatrix}^- \quad \begin{vmatrix} x_4 & x_1 & x_2 \\ y_4 & y_1 & y_2 \\ z_4 & z_1 & z_2 \end{vmatrix}^+$$



$$\begin{vmatrix} x_4 & x_2 & x_3 & x_1 & x_4 & x_2 \\ y_4 & y_2 & y_3 & y_1 & y_4 & y_2 \\ z_4 & z_2 & z_3 & z_1 & z_4 & z_2 \end{vmatrix}$$

$$\begin{vmatrix} x_4 & x_2 & x_3 \\ y_4 & y_2 & y_3 \\ z_4 & z_2 & z_3 \end{vmatrix}^+ \quad \begin{vmatrix} x_2 & x_3 & x_1 \\ y_2 & y_3 & y_1 \\ z_2 & z_3 & z_1 \end{vmatrix}^-$$

$$\begin{vmatrix} x_3 & x_1 & x_4 \\ y_3 & y_1 & y_4 \\ z_3 & z_1 & z_4 \end{vmatrix}^+ \quad \begin{vmatrix} x_1 & x_4 & x_2 \\ y_1 & y_4 & y_2 \\ z_1 & z_4 & z_2 \end{vmatrix}^-$$



$$\begin{vmatrix} x_1 & x_2 & x_4 & x_3 & x_1 & x_2 \\ y_1 & y_2 & y_4 & y_3 & y_1 & y_2 \\ z_1 & z_2 & z_4 & z_3 & z_1 & z_2 \end{vmatrix}$$

$$\begin{vmatrix} x_1 & x_2 & x_4 \\ y_1 & y_2 & y_4 \\ z_1 & z_2 & z_4 \end{vmatrix}^+ \quad \begin{vmatrix} x_2 & x_4 & x_3 \\ y_2 & y_4 & y_3 \\ z_2 & z_4 & z_3 \end{vmatrix}^-$$

$$\begin{vmatrix} x_4 & x_3 & x_1 \\ y_4 & y_3 & y_1 \\ z_4 & z_3 & z_1 \end{vmatrix}^+ \quad \begin{vmatrix} x_3 & x_1 & x_2 \\ y_3 & y_1 & y_2 \\ z_3 & z_1 & z_2 \end{vmatrix}^-$$

Volume Computation

f_1

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}^{-} + \begin{vmatrix} x_2 & x_3 & x_1 \\ y_2 & y_3 & y_1 \\ z_2 & z_3 & z_1 \end{vmatrix}^{-} + \begin{vmatrix} x_3 & x_1 & x_2 \\ y_3 & y_1 & y_2 \\ z_3 & z_1 & z_2 \end{vmatrix}^{-} = \det\begin{pmatrix} x_1 & x_3 & x_2 \\ y_1 & y_3 & y_2 \\ z_1 & z_3 & z_2 \end{pmatrix}$$

f_2

$$\begin{vmatrix} x_2 & x_3 & x_4 \\ y_2 & y_3 & y_4 \\ z_2 & z_3 & z_4 \end{vmatrix}^{+} + \begin{vmatrix} x_4 & x_2 & x_3 \\ y_4 & y_2 & y_3 \\ z_4 & z_2 & z_3 \end{vmatrix}^{+} + \begin{vmatrix} x_2 & x_4 & x_3 \\ y_2 & y_4 & y_3 \\ z_2 & z_4 & z_3 \end{vmatrix}^{-} = \det\begin{pmatrix} x_4 & x_2 & x_3 \\ y_4 & y_2 & y_3 \\ z_4 & z_2 & z_3 \end{pmatrix}$$

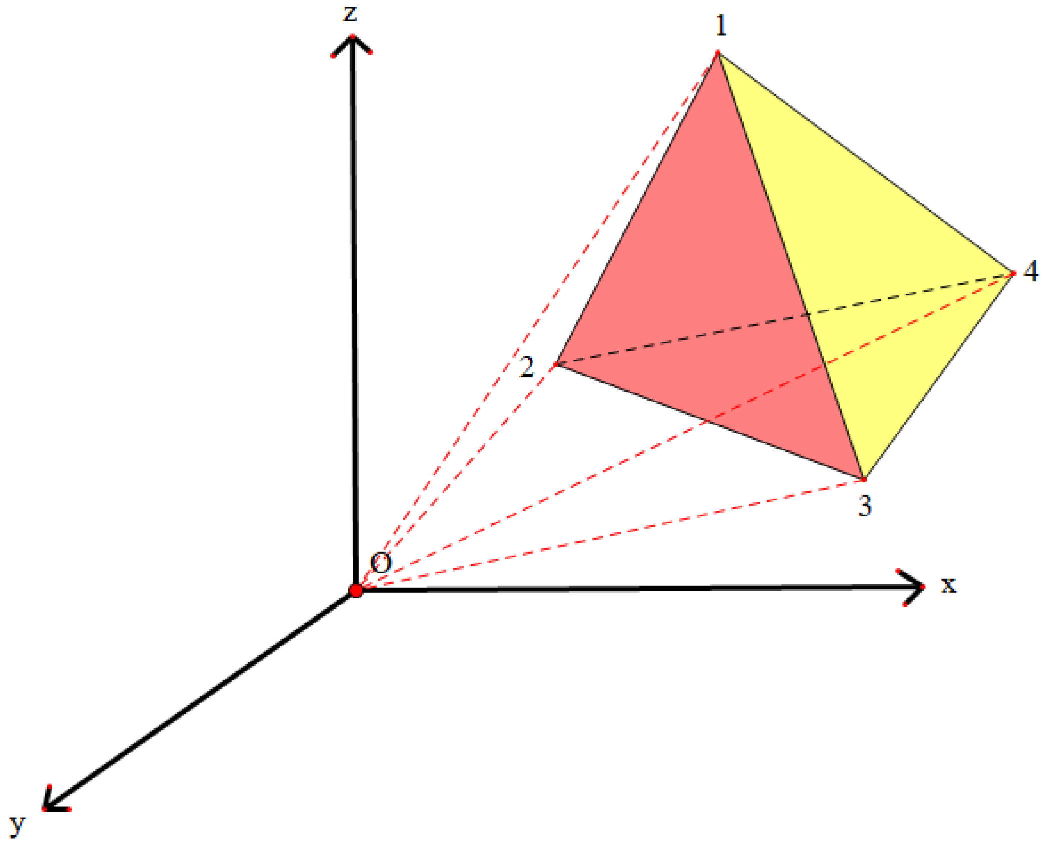
f_3

$$\begin{vmatrix} x_3 & x_4 & x_1 \\ y_3 & y_4 & y_1 \\ z_3 & z_4 & z_1 \end{vmatrix}^{-} + \begin{vmatrix} x_3 & x_1 & x_4 \\ y_3 & y_1 & y_4 \\ z_3 & z_1 & z_4 \end{vmatrix}^{+} + \begin{vmatrix} x_4 & x_3 & x_1 \\ y_4 & y_3 & y_1 \\ z_4 & z_3 & z_1 \end{vmatrix}^{+} = \det\begin{pmatrix} x_3 & x_1 & x_4 \\ y_3 & y_1 & y_4 \\ z_3 & z_1 & z_4 \end{pmatrix}$$

f_4

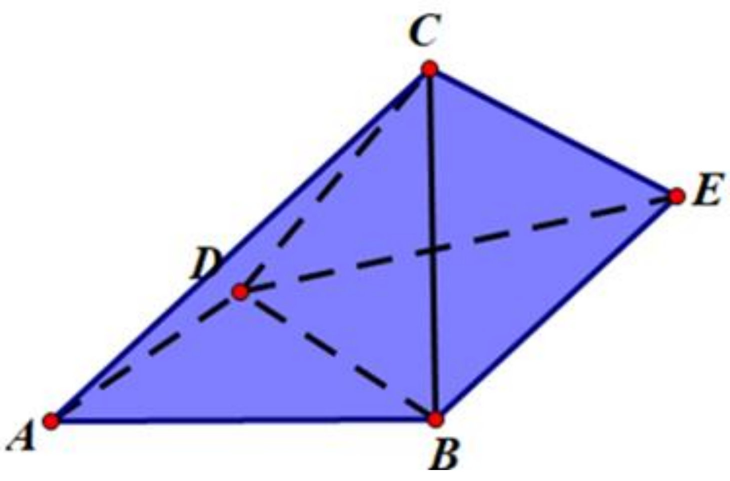
$$\begin{vmatrix} x_4 & x_1 & x_2 \\ y_4 & y_1 & y_2 \\ z_4 & z_1 & z_2 \end{vmatrix}^{+} + \begin{vmatrix} x_1 & x_4 & x_2 \\ y_1 & y_4 & y_2 \\ z_1 & z_4 & z_2 \end{vmatrix}^{-} + \begin{vmatrix} x_1 & x_2 & x_4 \\ y_1 & y_2 & y_4 \\ z_1 & z_2 & z_4 \end{vmatrix}^{+} = \det\begin{pmatrix} x_4 & x_1 & x_2 \\ y_4 & y_1 & y_2 \\ z_4 & z_1 & z_2 \end{pmatrix}$$

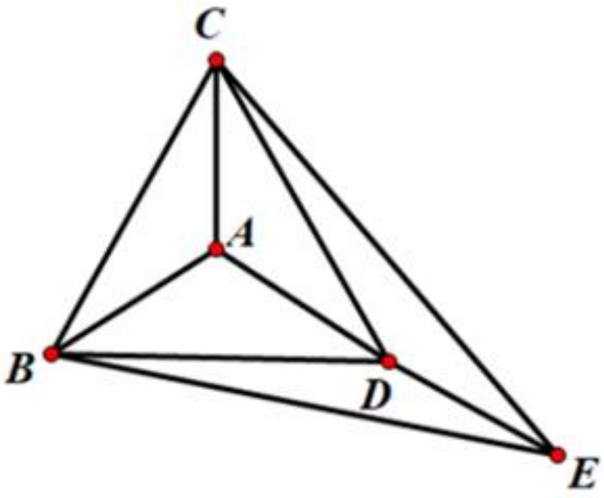
Volume Computation



Example of the Extended Shoelace Method

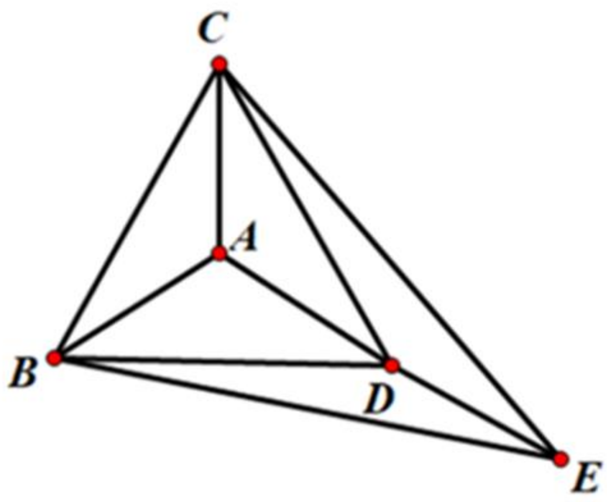
Step 1:
The given convex polyhedron is a bipyramid with coordinates A(-1,-1, 0) B(-1,1,0) C(-1, 1,2) D(1, 1, 0) E(0, 3, 1). Draw the MPG corresponding to it.





Example of the Extended Shoelace Method

Step 2:
Trace all non-equivalent Petrie walks on the MPG.



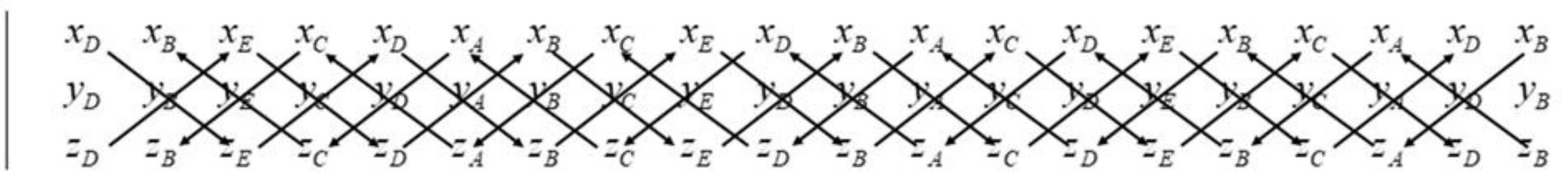
For this case ONLY, there exist only one Petrie Walk: DBECDABCEDBACDEBCA(D)

Example of the Extended Shoelace Method

Step 3:
Insert the coordinates of the vertices in the shoelace expression and calculate it.

Petrie Walk: DBECDABCEDBACDEBCA(D)

To insert the walk into the shoelace expression, we need to append a 'B' at the end



The Value of the expression is 16. Multiplying a factor of $\frac{1}{6}$, we get $\frac{8}{3}$.

Hence, the value volume of the bipyramid is: $\frac{8}{3}$

Example of the Extended Shoelace Method

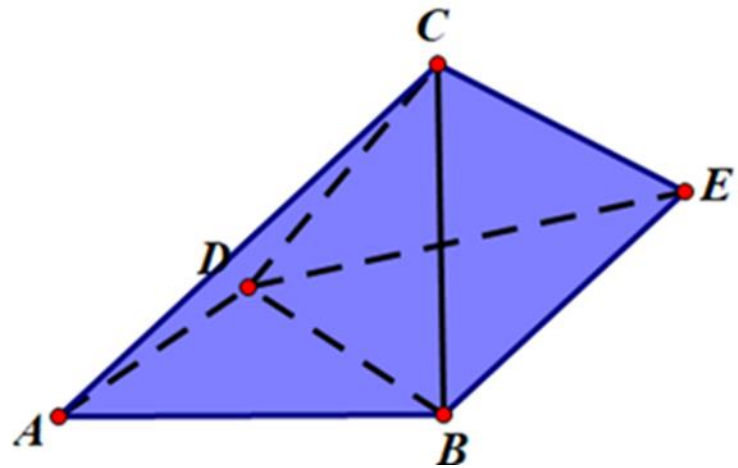
I will now use the volume formula of pyramids to verify our answer.

Since plane BCD is perpendicular to the x-y plane, we can use it as the base of the two pyramids.

$$\text{Volume of bipyramid} = V_{CDBE} + V_{CDBA} = \frac{1}{3} \times A_{\Delta CDB} \times (h_1 + h_2)$$

$$= \frac{1}{3} \times \left(\frac{1}{2} \times 2 \times 2\right) \times (2+2) = \frac{8}{3}$$

Which agrees with our result.



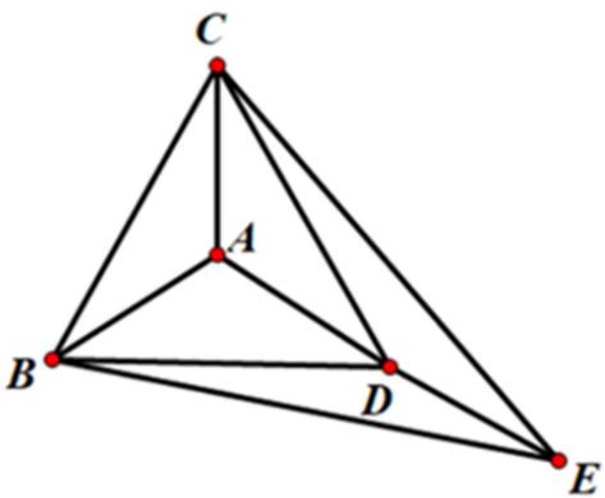
Conclusion

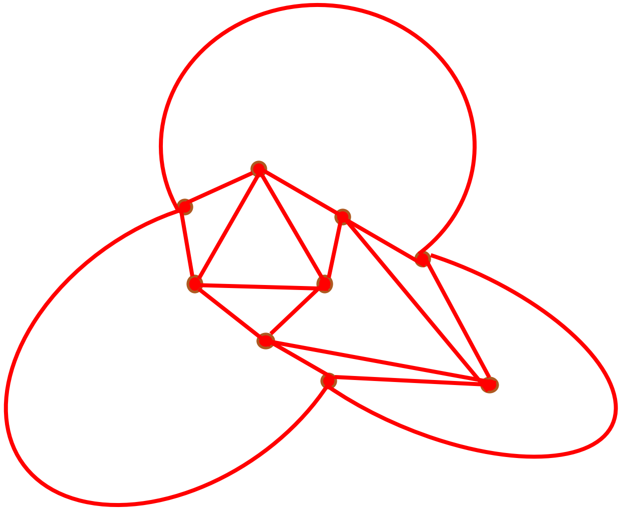
Conclusion

- I introduced Petrie Walks, which can line out CEPs in an MPG.
- I also discussed the equivalence of two Petrie Walks
- I mentioned a special type of MPG: Petrie-Complete MPGs, and its verification method (length of Petrie Walk= 6N-12)

Conclusion

- I found a one-to-one relation between MPGs and their Medial Graphs and found a correspondence between PCMPGs and Medial Graphs with Eulerian Circuits.





Conclusion

- Finally, since the determinant of a 3 by 3 matrix is equal to the tetrahedron formed, I am able to insert Petrie Walks into shoelace expressions.

- Recall my objective: "to establish the relationship between the result of shoelace method on 3-dimension figure to its volume."

- It has been accomplished with this result:

Volume of Polyhedron= $\frac{1}{6}\sum_{i=1}^{|W(G)|} \|w_i\|$

Future Work

- Extend the input of our algorithm to all polyhedra
- Look for trends in the Shoelace Method in even higher dimensions

Acknowledgement

- I would like to thank my school, NUS High School of Math and Science, for its continuous support throughout my journey.

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References

- Steinitz, Ernst. “Polyeder und Raumeinteilungen” *Encyk der Math Wiss 12* (1922): 38-43
- Shank, H. "The theory of left-right paths." Combinatorial Mathematics III. Springer, Berlin, Heidelberg, 1975. 42-54
- Deza, Michel, Mathieu Dutour, and Patrick W. Fowler. "Zigzags, railroads, and knots in fullerenes." Journal of chemical information and computer sciences 44.4 (2004): 1282-1293.
- Richter, R. "Spanning Trees, Euler tours, medial graphs, left-right paths and cycle spaces". Discrete Mathematics 89 (1991): 261-268.

Thank you for listening!